

Asymptotically warped anti-de Sitter spacetimes in topologically massive gravity

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Abstract

Asymptotically warped AdS spacetimes in topologically massive gravity with negative cosmological constant are considered in the case of spacelike stretched warping, where black holes have been shown to exist. We provide a set of asymptotic conditions that accommodate solutions in which the local degree of freedom (the “massive graviton”) is switched on. An exact solution with this property is explicitly exhibited and possesses a slower fall-off than the warped AdS black hole. The boundary conditions are invariant under the semidirect product of the Virasoro algebra with a $u(1)$ current algebra. We show that the canonical generators are integrable and finite. When the graviton is not excited, our analysis is compared and contrasted with earlier results obtained through the covariant approach to conserved charges. In particular, we find agreement with the conserved charges of the warped AdS black holes as well as with the central charges in the algebra.

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I. INTRODUCTION

Warped AdS spacetimes [1–3] have attracted considerable interest recently as possible ground states of topologically massive gravity with a negative cosmological constant [4]. They also become relevant in the context of Kerr/CFT correspondence since they emerge in the near horizon geometry of extremal Kerr black holes [5].

Topologically massive gravity with a negative cosmological constant [6] is described by the action [7]

$$I[e] = 2 \int \left[e^a \left(d\omega_a + \frac{1}{2} \epsilon_{abc} \omega^b \omega^c \right) + \frac{1}{6} \frac{1}{\ell^2} \epsilon_{abc} e^a e^b e^c \right] + \frac{1}{\mu} \int \omega^a \left(d\omega_a + \frac{1}{3} \epsilon_{abc} \omega^b \omega^c \right) \quad (1)$$

where $\mu \neq 0$ is the mass parameter, and l is the AdS radius. Hereafter we will focus on the case $|\mu l| > 3$ for which black holes with spacelike stretched warping have been shown to exist, [4, 8].

Asymptotically warped AdS spacetimes have already been explored in Refs. [9, 10]. The study provided there can be regarded as the analog of the boundary analysis of [11] for asymptotically AdS spacetimes.

These analyses cover black holes. However, topologically massive gravity has one local degree of freedom, which one can call the “graviton”. In order to accommodate solutions in which this local degree of freedom is excited, it was shown in [12, 13] that the boundary conditions of [11] for asymptotically AdS spacetimes need to be relaxed. It turns out that a similar relaxation is also necessary in the context of asymptotically warped AdS spacetimes. Indeed, as shown here, there exist solutions describing a propagating massive graviton on a warped AdS black hole, and these do not fulfill the boundary conditions of [10], [14].

We provide in this paper a set of relaxed asymptotic conditions that accommodate these solutions in which the local degree of freedom is switched on. Just as in the absence of the graviton [10], these boundary conditions are invariant under the semidirect product of the Virasoro algebra with a $u(1)$ current algebra. We compute the charges associated with these symmetries and show that they are finite, integrable, and as expected from the theorem of [18], they fulfill a centrally extended version of the asymptotic symmetry algebra.

The plan of the paper is as follows. In the next Section we provide an exact solution describing a propagating massive graviton on a warped AdS black hole. The suitable boundary conditions that accommodate this class of solutions are discussed in Sec. III. We then study

the asymptotic symmetry algebra (Section IV) and study its canonical realization in Sec. V. Next we study the restricted boundary conditions obtained by switching off the graviton and we compare and contrast them with the earlier results of [10] and [19]. In particular, we find in that case agreement with the conserved charges of the warped AdS black holes as well as with the central charges in the algebra. The last Section is devoted to conclusions. Some technical developments are given in the appendix.

II. NEW SOLUTION: WARPED ADS BLACK HOLE WITH A BOUNCING GRAVITON

Our starting point are the warped AdS black holes [8], given by

$$d\bar{s}^2 = dt^2 + \frac{l^2}{3 + \nu^2} \frac{dr^2}{(r - r_+)(r - r_-)} - 2 \left(\nu r + \frac{1}{2} \sqrt{r_+ r_- (3 + \nu^2)} \right) dt d\phi \quad (2)$$

$$+ \frac{r}{4} \left[3(\nu^2 - 1)^2 r + (3 + \nu^2)(r_+ + r_-) + 4\nu \sqrt{r_+ r_- (3 + \nu^2)} \right] d\phi^2 ,$$

which have been expressed in the coordinates of [4]. Note in particular that ϕ is an angle, $0 \leq \phi \leq 2\pi$. In (2), ν is equal to

$$\nu = \frac{\mu l}{3} .$$

If one sets $r_+ = r_- = 0$ one gets the zero mass black hole, whose metric explicitly reads,

$$ds_0^2 = dt^2 + \frac{l^2}{3 + \nu^2} \frac{dr^2}{r^2} - 2\nu r dt d\phi + \frac{3}{4}(\nu^2 - 1)^2 r^2 d\phi^2 . \quad (3)$$

The terms containing r_+ and r_- are subleading so that all the metrics of the above form share the same asymptotics.

Stretched warped AdS can be recovered from (3) once the corresponding coordinate ϕ is unwrapped. For this reason, it is customary to say that the solutions (2) are asymptotic to warped AdS although they are only so locally since ϕ is an angle in (2) while it is actually unwrapped in spacelike stretched warped AdS spacetime. In this paper we will adopt the same abuse of language.

In order to describe a massive graviton propagating on an arbitrary warped AdS black hole at the full non linear level, we follow the Kerr-Schild method [20]. We thus look for a solution of the form

$$ds^2 = d\bar{s}^2 + A(r, t, \phi) k_\mu k_\nu dx^\mu dx^\nu , \quad (4)$$

where

$$k = k^\mu \partial_\mu ,$$

is a null geodesic vector of $d\bar{s}^2$; i.e., $k^\mu k_\mu = 0$, and $k^\mu \bar{\nabla}_\mu k^\nu = 0$. We choose the one-form $k = k_\mu dx^\mu$ to be

$$k = \frac{2}{3 + \nu^2} \frac{l^2}{(r - r_+)(r - r_-)} dr + l d\phi . \quad (5)$$

Even though k^μ is not a Killing vector, the field equations can be readily integrated because they become linear in the unknown function A . A particular class of solutions interesting for our purposes is given by

$$A = e^{-\omega t} \left(\frac{(r - r_-)^{2\nu r_- + \sqrt{r_+ r_- (3 + \nu^2)}}}{(r - r_+)^{2\nu r_+ + \sqrt{r_+ r_- (3 + \nu^2)}}} \right)^{\frac{\omega l}{(r_+ - r_-)(3 + \nu^2)}} F \left[\frac{2l}{(r_+ - r_-)(3 + \nu^2)} \log \left(\frac{r - r_-}{r - r_+} \right) - \phi \right] , \quad (6)$$

where F is an arbitrary function, and

$$\omega l = \omega_\pm l = -3\nu \pm \sqrt{4\nu^2 - 3} . \quad (7)$$

Note that the solution remains a solution if one simultaneously change $t \rightarrow -t$, and $\phi \rightarrow -\phi$, a coordinate transformation that preserves the spacetime orientation. This coordinate change amounts to making $\omega \rightarrow -\omega$ in the solution.

A detailed physical interpretation of this solution will be given in [21], where the behaviour of the metric as $t \rightarrow \pm\infty$ and the isometries will be systematically investigated. It will also be shown in that work that a similar solution constructed from a BTZ black hole [22] as background “seed” metric exists.

For the purposes of this paper, the interest of the metric (4) with k_μ and A given by (5) and (6), respectively, is that it motivates the boundary conditions below. Note that the branch $\omega = \omega_-$ is such that the curvature does not approach to the one of Warped AdS at infinity. For this reason it will be discarded from now on and we set $\omega \equiv \omega_+$.

III. BOUNDARY CONDITIONS

As we now show, the branch ω_+ is asymptotically warped AdS in a relaxed sense as compared with the results of [10].

The asymptotic conditions are written with respect to the background, which is chosen as the warped AdS black hole with $r_+ = r_- = 0$. Thus, the asymptotic form of the metric is given by $ds^2 = ds_0^2 + \Delta g_{\mu\nu} dx^\mu dx^\nu$.

We adopt as boundary conditions

$$\begin{aligned}
\Delta g_{rr} &= h_{rr}^{(1)} r^{\alpha-4} + f_{rr} r^{-3} + Y_{rr} r^{2\alpha-6} + h_{rr}^{(2)} r^{\alpha-5} + c_{rr} r^{-4} + \dots \\
\Delta g_{rt} &= h_{rt}^{(1)} r^{\alpha-4} + f_{rt} r^{-3} + \dots \\
\Delta g_{r\phi} &= h_{r\phi}^{(1)} r^{\alpha-2} + Y_{r\phi} r^{2\alpha-4} + h_{r\phi}^{(2)} r^{\alpha-3} + f_{r\phi} r^{-2} + \dots \\
\Delta g_{tt} &= O(r^{-3}) \\
\Delta g_{t\phi} &= h_{t\phi}^{(1)} r^{\alpha-1} + f_{t\phi} + Y_{t\phi} r^{2\alpha-3} + h_{t\phi}^{(2)} r^{\alpha-2} + c_{t\phi} r^{-1} + \dots \\
\Delta g_{\phi\phi} &= h_{\phi\phi}^{(1)} r^\alpha + f_{\phi\phi} r + Y_{\phi\phi} r^{2\alpha-2} + h_{\phi\phi}^{(2)} r^{\alpha-1} + c_{\phi\phi} + \dots
\end{aligned} \tag{8}$$

where the h 's, Y 's, f 's and c 's are arbitrary functions of time and the angle. α is an arbitrary fixed constant which must be smaller than two, and which for simplicity we assume to be in the range

$$0 < \alpha < \frac{3}{2}, \quad \alpha \neq 1.$$

If α were not in this range there would be additional terms in the expansion. Note in particular that the cases $\alpha = 0, 1$ require special care because of the potential presence of logarithmic terms.

The functions h 's and Y 's are present only when the graviton is switched on. They are zero when it is not excited, a simplified context that will be studied in section VI below.

The boundary conditions have been motivated by an examination of the asymptotic behaviour of the above solution which has $\alpha = -2\nu \frac{\omega l}{3+\nu^2}$, with ω given by ω_+ in Eq. (7). Note that for the particular solution above, $\Delta g_{t\phi}$ vanishes. We have included a nontrivial $\Delta g_{t\phi}$ in the asymptotic conditions as this is required by the consistency of the asymptotic analysis carried out below.

The boundary conditions (8) must be completed by additional constraints on the coefficients appearing in the expansion in powers of r in the asymptotic form of the metric. As we will see below, these coefficients must be subject to some conditions that ensure finiteness of the canonical generators. These conditions are relatively intricate and for that reason are written in the appendix. These conditions are fulfilled by the exact solution.

We are now going to verify the consistency of the asymptotic conditions. This involves two steps. First, the boundary conditions should be invariant under the expected asymptotic

symmetry algebra, namely the semi direct product of the Virasoro algebra (V) with the $u(1)$ current algebra, i.e., $V \times_{\sigma} u(1)^+$. Second, the surface integrals appearing in the canonical generators should be finite. We verify these crucial existence requirements in the next two sections.

IV. ASYMPTOTIC SYMMETRY ALGEBRA

The asymptotic conditions (8) are mapped into themselves under the following asymptotic Killing vectors

$$\begin{aligned}\eta^t &= T + \frac{4\nu l^2}{(3 + \nu^2)^2} \frac{1}{r} \partial_{\phi}^2 X + \dots \\ \eta^{\phi} &= X + \frac{2l^2}{(3 + \nu^2)^2} \frac{1}{r^2} \partial_{\phi}^2 X + \dots \\ \eta^r &= -r \partial_{\phi} X + \dots\end{aligned}\tag{9}$$

where X and T are functions of ϕ only. These asymptotic transformations not only preserve the asymptotic form of the metric, but also the extra conditions on the coefficients that must be imposed by finiteness of the charges, as we pointed out, and which are written in the appendix.

The asymptotic Killing vectors close in the Lie bracket according to the semi direct product of the Virasoro algebra with the $u(1)$ current algebra, $V \times_{\sigma} u(1)^+$. The Virasoro algebra is parametrized by $X(\phi)$ while the $T(\phi)$ spans the affine Kac-Moody extension of $u(1)$. Explicitly, the commutation relations read $[\eta_1, \eta_2] = \eta_3$, with

$$\begin{aligned}X_3 &= X_1 \partial_{\phi} X_2 - X_2 \partial_{\phi} X_1 , \\ T_3 &= X_1 \partial_{\phi} T_2 - X_2 \partial_{\phi} T_1 .\end{aligned}$$

As it will be seen below, the subleading terms in (9) are crucial for getting a non vanishing central charge in the canonical realization of the algebra.

It is worth pointing out that the subset of the asymptotic conditions (8) for which the coefficients f 's and c 's are time independent is still consistent with the asymptotic symmetries. This is because $X(\phi)$ and $T(\phi)$ depend only on ϕ and not on t , and so do not introduce through the Lie action a time-dependence in the f 's and the c 's (which do not mix with the h 's and the Y 's) if there is none to begin with. Assuming the f 's and the c 's not to depend

on time – while allowing the “graviton-characterizing” coefficients h ’s and Y ’s to depend on time – does not exclude the above metric (4). Therefore, this is an interesting subset of the asymptotic conditions, which actually turns out to simplify dramatically the formulas for the charges. These formulas are in fact explicitly given below only in that case. This is sufficient, as our purposes are to accommodate the graviton on an arbitrary warped AdS black hole. The extra conditions on the metric coefficients necessary for ensuring finiteness of the charges are also spelled out in the appendix only in that case.

V. CANONICAL GENERATORS

A. Surface integrals

We compute the conserved charges within the canonical formalism, “à la Regge-Teitelboim” [23]. As explained there, the charges that generate the diffeomorphisms (9) take the form

$$H[\eta] = \text{“Bulk piece”} + Q[T] + Q[X], \quad (10)$$

where the bulk piece is a linear combination of the constraints with coefficients involving η^t, η^ϕ , and η^r , and where $Q[T]$ and $Q[X]$ are surface integrals at infinity that involve only the asymptotic form of the vector field η^t, η^ϕ , and η^r . On shell, the bulk piece vanishes and $H[\eta]$ reduces to $Q[T] + Q[X]$.

The canonical analysis of topologically massive gravity has been performed in [24], and here we follow the procedure devised in [25] and further developed in [13].

The general variation of the surface integrals was derived in [13]. Applying this to the above asymptotic behaviour, one gets variations of the surface integrals $\delta Q[T]$ and $\delta Q[X]$ which are finite thanks to the extra conditions on the metric coefficients. These variations are furthermore integrable without requiring further conditions.

As pointed out in the previous section, the subset of the asymptotic conditions (8) for which the coefficients f ’s and c ’s are time independent is consistent with the asymptotic symmetries. In this case, the corresponding charges are

$$Q[T] = \frac{2}{3l\nu} \int d\phi \, T \, [3(\nu^2 - 1)f_{t\phi} + 2\nu f_{\phi\phi}] , \quad (11)$$

and

$$\begin{aligned}
Q[X] = & \frac{1}{12l^3\nu} \frac{1}{(3+\nu^2)^2} \int d\phi \, X \left[-3(3+\nu^2)^3 (\nu^2-1)(2\nu^2-3) c_{rr} \right. \\
& + 2l^2 (9\nu(\nu^2-1)(3+\nu^2)(3+5\nu^2) c_{t\phi} + 2((3+\nu^2)(18+3\nu^2+11\nu^4) c_{\phi\phi} \\
& + 3(\nu^2-1)(27-39\nu^2+32\nu^4) f_{t\phi}^2 \\
& \left. + 8\nu(27-24\nu^2+13\nu^4) f_{t\phi} f_{\phi\phi} + (45-54\nu^2+25\nu^4) f_{\phi\phi}^2) \right] .
\end{aligned} \tag{12}$$

In these formulas we have adjusted the integration constants such that the charges vanish for the background configuration (3). The “ rr ” component of the field equations allows to express c_{rr} in terms of the rest of the coefficients appearing in $Q[X]$ (see appendix). This equation turns out to be invariant under the asymptotic symmetries. One can verify that on-shell, $Q[X]$ simplifies to

$$\begin{aligned}
Q[X] = & \frac{1}{6l\nu^2} \int d\phi \, X \left[(5\nu^2+3)(3(\nu^2-1)c_{t\phi} + 2\nu c_{\phi\phi}) \right. \\
& \left. + 12\nu(\nu^2-1)f_{t\phi}^2 + 16\nu^2 f_{t\phi} f_{\phi\phi} + 2\nu f_{\phi\phi}^2 \right] .
\end{aligned} \tag{13}$$

The same procedure applies to the general situation when the coefficients f ’s and c ’s depend on time: the charges exist and are finite but their expression is much more cumbersome.

The charges (11) and (12) (or (13)) possess the intriguing feature of not depending on the coefficients h ’s and the Y ’s characterizing the graviton. Thus, the graviton does not contribute directly to the asymptotic charges in topologically massive gravity with the above boundary conditions, even though it does enter non trivially these boundary conditions. A similar phenomenon was encountered in the study of asymptotically AdS spacetimes in topologically massive gravity [12, 13] (see also [26]).

B. Algebra of charges

According to the general theorem demonstrated in [18] the charges close in the Poisson bracket according to the algebra of the corresponding asymptotic symmetries, modulo possible central extensions. For that reason, the only task for getting the algebra of the charges is to compute the central extensions. To that end it is useful to observe that under an

asymptotic symmetry, the relevant coefficients in (11) and (13) transform as

$$\begin{aligned}
\delta f_{\phi\phi} &= f_{\phi\phi} \partial_\phi X + X \partial_\phi f_{\phi\phi} - 2\nu \partial_\phi T , \\
\delta f_{t\phi} &= f_{t\phi} \partial_\phi X + X \partial_\phi f_{t\phi} + \partial_\phi T , \\
3(\nu^2 - 1)\delta c_{t\phi} + 2\nu\delta c_{\phi\phi} &= 6(\nu^2 - 1)c_{t\phi}\partial_\phi X - \frac{9}{\nu^2 + 3}X\partial_\phi c_{t\phi} \\
&\quad + \nu \left(4c_{\phi\phi}\partial_\phi X + 4f_{t\phi}\partial_\phi T - \frac{4}{\nu^2 + 3}l^2\partial_\phi^3 X \right. \\
&\quad \left. + X \left(3\nu\frac{\nu^2 + 2}{\nu^2 + 3}\partial_\phi c_{t\phi} + 2\partial_\phi c_{\phi\phi} \right) \right) .
\end{aligned}$$

Using these expressions we immediately find the central charge and the Poisson bracket of the generators, which read explicitly as

$$[Q(\eta_1), Q(\eta_2)] = Q([\eta_1, \eta_2]) + K(\eta_1, \eta_2) ,$$

with

$$K(\eta_1, \eta_2) = -\frac{2}{3l\nu} \int d\phi \left[(\nu^2 + 3)T_1\partial_\phi T_2 + l^2\frac{5\nu^2 + 3}{\nu^2 + 3}X_1\partial_\phi^3 X_2 \right] .$$

In terms of the Fourier components, the algebra reads

$$\begin{aligned}
i[L_m, L_n] &= (m - n)L_{m+n} + \frac{c_V}{12} m^3 \delta_{m+n,0} , \\
i[L_m, T_n] &= -nT_{m+n} , \\
i[T_m, T_n] &= -\frac{c_{u(1)}}{12} m\delta_{m+n,0} ,
\end{aligned}$$

with

$$c_V = \frac{5\nu^2 + 3}{\nu(\nu^2 + 3)} \frac{l}{G} ; \quad c_{u(1)} = \frac{\nu^2 + 3}{\nu} \frac{l}{G} .$$

VI. SWITCHING OFF THE GRAVITON: RESTRICTED BOUNDARY CONDITIONS

When the graviton is not excited, the subset of boundary conditions (8) further simplifies to

$$\begin{aligned}
\Delta g_{rr} &= f_{rr} r^{-3} + c_{rr} r^{-4} + \dots \\
\Delta g_{rt} &= O(r^{-3}) \\
\Delta g_{r\phi} &= O(r^{-2}) \\
\Delta g_{tt} &= O(r^{-3}) \\
\Delta g_{t\phi} &= f_{t\phi} + c_{t\phi} r^{-1} + \dots \\
\Delta g_{\phi\phi} &= f_{\phi\phi} r + c_{\phi\phi} + \dots
\end{aligned} \tag{14}$$

and the condition on the coefficients that guarantees the finiteness of the charges just reads

$$(3 + \nu^2)^2 f_{rr} - 4l^2 (2\nu f_{t\phi} + f_{\phi\phi}) = 0 . \quad (15)$$

It is easy to check that the warped AdS black holes (2) fulfill all these restricted boundary conditions as well as the condition (15) on the coefficients.

Since the charges obtained in the previous section do not depend on the h 's and the Y 's, they take exactly the same form in this restricted setting. The algebra of the charges, and in particular the central charges, are of course unchanged.

As stated in the introduction, this restricted setting was already investigated in [10], and [19]. As can be checked by direct inspection, while our boundary conditions turn out to be different, we obtain the same central charges, but our algebra only agrees with the one in [19]. There is another important difference between our approach and the previous treatment of [10]. While this latter treatment uses the equations of motion, and so defines the charges only for on-shell configurations, our approach enables one to define the conserved charges even for off-shell configurations, satisfying the asymptotic conditions (14). This appears to be important for the path integral.

VII. SUMMARY AND FINAL REMARKS

In this paper the set of asymptotic conditions given by Eq. (8) has been shown to accommodate asymptotically warped AdS solutions in the case of spacelike stretched warping where the local degree of freedom is switched on. An explicit exact solution of this sort, possessing a slower fall-off than the warped AdS black hole was provided in Section II. The relaxed set of boundary conditions turn out to be invariant under the semidirect product of the Virasoro algebra with a $u(1)$ current algebra. The canonical generators were found to be integrable and finite, while the subleading terms in (9) were shown to play a key rôle in order to obtain a non vanishing central charge.

Remarkably, the subset of the asymptotic conditions (8), for which the f and the c coefficients are time independent, is still consistent with the asymptotic symmetries, which allows to simplify the form of the canonical generators as in Eqs. (11) and (13).

As it occurs for asymptotically AdS spacetimes in topologically massive gravity [12, 13] the charges have the intriguing feature of not depending on the relaxation terms switched on

by the graviton [27]. Therefore, the global charges associated to the new solution presented here, reduce to the ones of the “seed” warped AdS black hole metric (2), i.e.,

$$M = Q[\partial_t] = \frac{3 + \nu^2}{24lG} \left(r_+ + r_- + \nu^{-1} \sqrt{r_+ r_- (3 + \nu^2)} \right) ,$$

and

$$J = Q[\partial_\phi] = \nu \frac{3 + \nu^2}{96lG} \left[\frac{5\nu^2 + 3}{4\nu^2} (r_+ - r_-)^2 - \left(r_+ + r_- + \nu^{-1} \sqrt{r_+ r_- (3 + \nu^2)} \right)^2 \right] .$$

This agrees with the results of [10] and [19] for the mass and the angular momentum of the warped AdS black hole in the absence of the graviton [32].

A similar construction can be performed for asymptotically AdS spacetimes in topologically massive gravity, where an analog solution constructed out from a BTZ black hole as background metric can be shown to exist. Further details about this solution, and about the similar effect of non-appearance of the graviton-characterizing asymptotic coefficients in the charges, will be discussed in [21].

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Appendix A: Conditions for the finiteness of the charges

In this appendix, we consider only the subset of the asymptotic conditions (8), for which the f 's and c 's coefficients are time independent. This is done for the sole purpose of

simplicity, in order to get formulas which, even though formidable, remain tractable. The general case can be treated along similar lines but is even more formidable.

The coefficients appearing in the expansion in powers of r in the asymptotic form of the metric (8) are subject to certain conditions that ensure the finiteness of the canonical generators. This can be seen as follows. For asymptotic conditions of the form (8) the corresponding charges spanned by the asymptotic symmetries (9) contain a finite piece, given by $Q[T]$ and $Q[X]$ in Eqs. (11) and (12), respectively, as well as some terms that diverge as $r \rightarrow \infty$ according to

$$Q[T, X] = Q[T] + Q[X] + \frac{1}{8l^3\nu(3+\nu^2)} \int d\phi \left[X \left(\frac{r^\alpha}{3} \mathcal{X}_{(\alpha)} - (\nu^2 - 1)(5\nu^2 - 3) r \mathcal{X}_{(1)} \right. \right. \\ \left. \left. + \frac{r^{2\alpha-2} \mathcal{X}_{(2\alpha-2)} + r^{\alpha-1} \mathcal{X}_{(\alpha-1)}}{3l^2(3+\nu^2)} \right) + \frac{4T}{3} r^{\alpha-1} \mathcal{T}_{(\alpha-1)} \right]. \quad (\text{A1})$$

Therefore, the canonical generators become finite for an arbitrary asymptotic symmetry provided $\mathcal{X}_{(i)}$ and $\mathcal{T}_{(\alpha-1)}$ vanish.

The simplest condition is independent of α and corresponds to the vanishing of the divergence that is linear in r , i.e., $\mathcal{X}_{(1)} = 0$ which only involves the f' 's and it reduces to Eq. (15). The leading divergence, r^a , gives a linear condition that depends on the $h^{(1)'}s$ and their derivatives, which reads

$$\mathcal{X}_{(\alpha)} = -3(\nu^2 - 1)(3 + \nu^2)^2(3(-2 + \alpha) + (4 + \alpha)\nu^2) h_{rr}^{(1)} \\ + l^2 \left(12\nu(\nu^2 - 1)(9 + 15\nu^2 + \alpha(-7 + 2\alpha)(3 + \nu^2)) h_{t\phi}^{(1)} \right. \\ + 4(9(-2 + \alpha)^2 + 6(1 + \alpha(-8 + 3\alpha))\nu^2 + (22 + \alpha(-12 + 5\alpha))\nu^4) h_{\phi\phi}^{(1)} \\ \left. + 3(\nu^2 - 1)(-4(-3 + 2\alpha)\nu(3 + \nu^2) \partial_t h_{r\phi}^{(1)} + 3(\nu^2 - 1)(\nu^2 + 3) \partial_t^2 h_{rr}^{(1)} - 4l^2 \partial_t^2 h_{\phi\phi}^{(1)}) \right). \quad (\text{A2})$$

There are two conditions coming from the divergence that go like $r^{\alpha-1}$, given by

$$\mathcal{T}_{(\alpha-1)} = 6\nu^3(3 + \nu^2)^2 h_{rr}^{(1)} - 6l^2(2\nu^2(3 + 5\nu^2) + \alpha(3 + \nu^2)(3 - 5\nu^2 + \alpha(\nu^2 - 1))) h_{t\phi}^{(1)} \\ - 4l^2\nu(6\nu^2 + \alpha(3 + \nu^2)(\alpha - 3)) h_{\phi\phi}^{(1)} + 6l^2(3 + \nu^2)(\alpha(\nu^2 - 1) - 2\nu^2) \partial_t h_{r\phi}^{(1)} \quad (\text{A3}) \\ - 3l^2\nu(\nu^2 - 1)(\nu^2 + 3) \partial_t^2 h_{rr}^{(1)} + 4l^4\nu \partial_t^2 h_{\phi\phi}^{(1)},$$

involving only and the $h^{(1)'}s$ and their derivatives, as well as a nonlinear one that reads

$$\begin{aligned}
\mathcal{X}_{(\alpha-1)} = & (3 + \nu^2)^2 f_{rr} \left(3 (-1 + \nu^2) (3 + \nu^2)^2 (6(-3 + \alpha) + (9 + 2\alpha)\nu^2) h_{rr}^{(1)} \right. \\
& - l^2 \left(6\nu (-1 + \nu^2) (3 + 37\nu^2 + \alpha(-13 + 4\alpha) (3 + \nu^2)) h_{t\phi}^{(1)} \right. \\
& + 2 (9(8 + \alpha(-9 + 2\alpha)) + 6(-3 + \alpha(-13 + 6\alpha))\nu^2 + (58 + \alpha(-17 + 10\alpha))\nu^4) h_{\phi\phi}^{(1)} \\
& \left. - 6(-5 + 4\alpha)\nu (-3 + 2\nu^2 + \nu^4) \partial_t h_{rr}^{(1)} + 9 (-1 + \nu^2)^2 (3 + \nu^2) \partial_t^2 h_{rr}^{(1)} \right) \\
& + l^2 \left(2f_{\phi\phi} \left(3 (-1 + \nu^2) (3 + \nu^2)^2 (3\alpha + (-18 + \alpha)\nu^2) h_{rr}^{(1)} \right. \right. \\
& - 2l^2 \left(2\nu (9(-23 + 5\alpha) + 6(5 + (35 - 8\alpha)\alpha)\nu^2 + (-207 + (65 - 16\alpha)\alpha)\nu^4) h_{t\phi}^{(1)} \right. \\
& - 4 (-18(-2 + \alpha) + 3(-9 + 2\alpha(-5 + 2\alpha))\nu^2 + (47 + 4(-2 + \alpha)\alpha)\nu^4) h_{\phi\phi}^{(1)} \\
& + 2\nu (3 + \nu^2) (-27 + (-21 + 16\alpha)\nu^2) \partial_t h_{r\phi}^{(1)} + 3 (9 + 9\nu^2 - 13\nu^4 - 5\nu^6) \partial_t^2 h_{rr}^{(1)} + 16l^2\nu^2 \partial_t^2 h_{\phi\phi}^{(1)} \Big) \\
& + 4f_{t\phi} \left(3\nu (-1 + \nu^2) (3 + \nu^2)^2 (3(3 + \alpha) + (-15 + \alpha)\nu^2) h_{rr}^{(1)} \right. \\
& + 2l^2 \left(3 (-1 + \nu^2) (9 (6 + \alpha - \alpha^2) + 6(-2 + \alpha(-14 + 3\alpha))\nu^2 + (110 + \alpha(-29 + 7\alpha))\nu^4) h_{t\phi}^{(1)} \right. \\
& + 2\nu (99 - 18\alpha^2 + 12(-6 + (-3 + \alpha)\alpha)\nu^2 + (85 + 6(-2 + \alpha)\alpha)\nu^4) h_{\phi\phi}^{(1)} \\
& - 3 (-3(2 + \alpha) + (-8 + 7\alpha)\nu^2) (-1 + \nu^2) (3 + \nu^2) \partial_t h_{r\phi}^{(1)} \\
& \left. + 9\nu (-1 + \nu^2)^2 (3 + \nu^2) \partial_t^2 h_{rr}^{(1)} - 12l^2\nu (-1 + \nu^2) \partial_t^2 h_{\phi\phi}^{(1)} \right) \Big) \\
& - 3 (-1 + \nu^2) (3 + \nu^2)^3 (3(-3 + \alpha) + (3 + \alpha)\nu^2) h_{rr}^{(2)} \\
& + 12l^2\nu (-1 + \nu^2) (3 + \nu^2) (36 + 24\nu^2 + \alpha(-11 + 2\alpha) (3 + \nu^2)) h_{t\phi}^{(2)} \\
& + 4l^2 (3 + \nu^2) (9(-3 + \alpha)^2 + 6(12 + \alpha(-14 + 3\alpha))\nu^2 + (39 + \alpha(-22 + 5\alpha))\nu^4) h_{\phi\phi}^{(2)} \\
& - 8l^2(-3 + \alpha) (3 + \nu^2)^2 (3 + 5\nu^2) \partial_\phi h_{r\phi}^{(1)} - 12l^2(-5 + 2\alpha)\nu (-1 + \nu^2) (3 + \nu^2)^2 \partial_t h_{r\phi}^{(2)} \\
& - 18l^2(-4 + \alpha) ((-1 + \nu^2) (3 + \nu^2))^2 \partial_t h_{tr}^{(1)} + 12l^2\nu (-1 + \nu^2) (3 + \nu^2)^2 \partial_t \partial_\phi h_{rr}^{(1)} \\
& + 24l^4 (-1 + \nu^2) (3(-2 + \alpha) + (2 + \alpha)\nu^2) \partial_t \partial_\phi h_{t\phi}^{(1)} + 16l^4\nu (2\alpha (3 + \nu^2) - 3(7 + \nu^2)) \partial_t \partial_\phi h_{\phi\phi}^{(1)} \\
& \left. + 9l^2 (9 - 12\nu^2 - 2\nu^4 + 4\nu^6) \partial_t^2 h_{rr}^{(2)} - 12l^4 (-1 + \nu^2) (3 + \nu^2) \left(\partial_t^2 h_{\phi\phi}^{(2)} + 2\partial_t^2 \partial_\phi h_{r\phi}^{(1)} \right) \right).
\end{aligned} \tag{A4}$$

The remaining condition, coming from the divergence of order $O(r^{2\alpha-2})$, is a combination

of linear terms in the Y 's and quadratic involving the $h^{(1)'}s$ and it reads

$$\begin{aligned}
\mathcal{X}_{(2\alpha-2)} = & \frac{3}{2} (\nu^2 - 1) (3 + \nu^2)^4 (-24 + 7\nu^2 + 4\alpha (3 + \nu^2)) (h_{rr}^{(1)})^2 \\
& - 24l^2 (\nu^2 - 1) (3 + \nu^2)^2 (-6 + \nu^2 + \alpha (3 + \nu^2)) (h_{r\phi}^{(1)})^2 \\
& + 12l^4 (\nu^2 - 1) (63 + 84\nu^2 + 141\nu^4 - 76\alpha\nu^2 (3 + \nu^2) + \alpha^2 (-9 + 66\nu^2 + 23\nu^4)) (h_{t\phi}^{(1)})^2 \\
& + 8l^4 (72 + 15\nu^2 + 57\nu^4 + 2\alpha^2 (9 + 24\nu^2 + 7\nu^4) - 4\alpha (18 + 27\nu^2 + 7\nu^4)) (h_{\phi\phi}^{(1)})^2 \\
& - 6l^2 (\nu^2 - 1) (3 + \nu^2)^3 (-6 + \nu^2 + \alpha (3 + \nu^2)) Y_{rr} \\
& + 12l^4 \nu (-3 + 2\nu^2 + \nu^4) (75 + 37\nu^2 - 30\alpha (3 + \nu^2) + 8\alpha^2 (3 + \nu^2)) Y_{t\phi} \\
& + 8l^4 (3 + \nu^2) (72 + 87\nu^2 + 33\nu^4 - 8\alpha (9 + 15\nu^2 + 4\nu^4) + 2\alpha^2 (9 + 18\nu^2 + 5\nu^4)) Y_{\phi\phi} \\
& - 12l^2 \nu (\nu^2 - 1) (3 + \nu^2)^3 h_{r\phi}^{(1)} \partial_t h_{rr}^{(1)} - \frac{9}{2} l^2 (\nu^2 - 1)^2 (3 + \nu^2)^3 (\partial_t h_{rr}^{(1)})^2 \\
& + 6l^2 (-7 + 6\alpha) \nu (\nu^2 - 1) (3 + \nu^2)^3 h_{rr}^{(1)} \partial_t h_{r\phi}^{(1)} + 12l^4 (\nu^2 - 1) (3 + \nu^2)^2 (\partial_t h_{rr}^{(1)})^2 \\
& + 12l^4 (-3 + 2\nu^2 + \nu^4) \left(-2 (15 - 7\nu^2 + 2\alpha (-3 + \nu^2)) h_{r\phi}^{(1)} + 3\nu (\nu^2 - 1) \partial_t h_{rr}^{(1)} \right) \partial_t h_{t\phi}^{(1)} \\
& - 2l^4 \left(-8\nu (3 + \nu^2) (-39 - \nu^2 + 3\alpha (7 + \nu^2)) h_{r\phi}^{(1)} \right. \\
& \left. - 3 (\nu^2 - 1) \left(3 (-3 + 2\nu^2 + \nu^4) \partial_t h_{rr}^{(1)} - 8l^2 \nu \partial_t h_{t\phi}^{(1)} \right) \right) \partial_t h_{\phi\phi}^{(1)} \\
& - 16l^6 (-3 + \nu^2) (\partial_t h_{\phi\phi}^{(1)})^2 - 9L^2 (\nu^2 - 1)^2 (3 + \nu^2)^3 h_{rr}^{(1)} \partial_t^2 h_{rr}^{(1)} \\
& - 6l^2 (-3 + 2\nu^2 + \nu^4) h_{t\phi}^{(1)} (\nu (3 + \nu^2) (15 + 41\nu^2 - 19\alpha (3 + \nu^2) + 6\alpha^2 (3 + \nu^2)) h_{rr}^{(1)} \\
& + 4l^2 \left((-2 (6 + 5\nu^2) + \alpha (3 + 9\nu^2)) \partial_t h_{r\phi}^{(1)} - 3\nu (\nu^2 - 1) \partial_t^2 h_{rr}^{(1)} \right) \\
& + 2l^2 h_{\phi\phi}^{(1)} \left(-3 (3 + \nu^2)^2 (-10\alpha (3 + 4\nu^2 + \nu^4) + \alpha^2 (9 + 18\nu^2 + 5\nu^4) + 2 (12 + \nu^2 + 11\nu^4)) h_{rr}^{(1)} \right. \\
& + 2l^2 \left(2\nu (2\alpha^2 (-9 + 66\nu^2 + 23\nu^4) + 3 (87 + 26\nu^2 + 79\nu^4) - \alpha (81 + 402\nu^2 + 125\nu^4)) h_{t\phi}^{(1)} \right. \\
& + (3 + \nu^2) \left(\nu (90 - 36\alpha + (54 - 44\alpha)\nu^2) \partial_t h_{r\phi}^{(1)} + 3 (-3 - 2\nu^2 + 5\nu^4) \partial_t^2 h_{rr}^{(1)} \right) \left. \right) \\
& - 24l^4 (\nu^2 - 1) (3 + \nu^2)^2 h_{r\phi}^{(1)} \partial_t^2 h_{r\phi}^{(1)} - 32l^6 \nu \left(3 (\nu^2 - 1) h_{t\phi}^{(1)} + 2\nu h_{\phi\phi}^{(1)} \right) \partial_t^2 h_{\phi\phi}^{(1)} \\
& + 9l^4 (-3 + 2\nu^2 + \nu^4)^2 \partial_t^2 Y_{rr} - 12l^6 (-3 + 2\nu^2 + \nu^4) \partial_t^2 Y_{\phi\phi} \\
& - 12l^4 \nu (3 + \nu^2)^2 (-1 + \nu^2) (4\alpha - 7) \partial_t Y_{r\phi}.
\end{aligned} \tag{A5}$$

Remarkably, the asymptotic symmetries (9) not only preserve the asymptotic form of the metric, but also these additional conditions on the coefficients in the expansion of the metric that are imposed by finiteness of the charges. Indeed, all but one of the conditions written above automatically fulfill

$$\mathcal{L}_\eta \mathcal{X}_{(i)} = 0 ; \mathcal{L}_\eta \mathcal{T}_{(\alpha-1)} = 0 .$$

A nontrivial consistency condition only comes from $\mathcal{L}_\eta \mathcal{X}_{(\alpha-1)}$, that reads

$$\begin{aligned} 0 = & 3(-1 + \nu^2)(3 + \nu^2)^2(3 + 5\nu^2 - 3\alpha(3 + \nu^2) + \alpha^2(3 + \nu^2))h_{rr}^{(1)} \\ & + l^2 \left(4(2 - \alpha)(3 + \nu^2)(3 + 5\nu^2 - 3\alpha(3 + \nu^2) + \alpha^2(3 + \nu^2))h_{\phi\phi}^{(1)} \right. \\ & \left. - 3(-1 + \nu^2) \left(3(-3 + 2\nu^2 + \nu^4)\partial_t^2 h_{rr}^{(1)} - 4l^2(-2 + \alpha)\partial_t^2 h_{\phi\phi}^{(1)} \right) \right), \end{aligned} \quad (\text{A6})$$

which nevertheless may be easily verified to be also invariant under the asymptotic symmetry group.

Besides, the “ rr ” component of the field equations that allows to simplify $Q[X]$ as in Eq. (13) is also invariant under the asymptotic symmetries, and reads

$$\begin{aligned} c_{rr} = & \frac{2l^2(3 + 5\nu^2)c_{t\phi}}{\nu(3 + \nu^2)^2} + \frac{1}{4l^2(-3 + \nu^2)(3 + \nu^2)^3} (16l^4(-9 + \nu^4)c_{\phi\phi} \\ & + 3(-1 + \nu^2)(3 + \nu^2)^4 f_{rr}^2 - 24l^2(-1 + \nu^2)(3 + \nu^2)^2 f_{rr}(2\nu f_{t\phi} + f_{\phi\phi}) \\ & - 16l^4(-3(3 - 20\nu^2 + 9\nu^4)f_{t\phi}^2 + (7\nu^2 - 15)f_{\phi\phi}(4\nu f_{t\phi} + f_{\phi\phi}))) . \end{aligned} \quad (\text{A7})$$

It is straightforward to verify that all of the conditions spelled above are fulfilled by the new exact solution in Section II.

As stated above, we have derived in this appendix the extra constraints on the coefficients in the metric expansion only for the subset of the asymptotic conditions (8) where the f 's and c 's coefficients are time independent. A tedious, but straightforward computation shows that when the f 's and c 's depend on time, the boundary conditions (8) are still consistent with the asymptotic symmetries in the sense explained here, but with even more intricate finiteness and consistency conditions.

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